A Force based Finite Element Method with Automated identification of the Redundant Forces.

Jakka Nagabhushanam (Retd. Professor)
Dept of Aerospace Engineering, Indian Institute of Science, Bangalore-560012
Email: jakkanaga@gmail.com
D. Roy Mahapatra(Associate Professor)
Dept of Aerospace Engineering, Indian Institute of Science, Bangalore-560012
Email: droymahapatra@aero.iisc.ac.in.com

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Abstract

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Generally, the stresses and forces of a structure are the prime parameters in the investigations of structural design and analysis. The classical force method was extensively used in the pre-computer era. This method did not gain its dominant place in the post-computer era due to the absence of a simple automated algorithms to classify the forces into independent and redundant forces and it is still eluding. On the other hand, the displacement method gained most popularity due to its simple and streamlined implementation on computers. Recently, the Integrated Force Method provided a formulation without the need classification of forces of the structure. However, the significance of the classification of forces cannot be ignored as it provides a path to extend the force method for many advanced investigations of the structures. This paper extends the Integrated Force Method to make it in line with the classical force method with the inclusion of automated identification of independent and redundant forces. It is independent of idealizations made on the structural elements of the finite element model. Thus it can bring back the past popular classical force method to suit the present post computer era. Its implementation of computers is as simple as the displacement method. Few illustrative examples of the complex structures

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idealized with the different elements are presented to demonstrates the successful identification of independent and redundant forces by the proposed algorithm.

1. Introduction.

The Finite Element Method of analysis of a structure finds the internal forces/stresses in its structural members and the displacements at the nodes. Two basic approaches are available to analyze these structures. They are the displacement method proposed by Clebsch(1833-1872) and Force Method by Maxwell(1831-1879). These methods have been in existence for close to two centuries.

In the pre-computer era, structural models were simple with equally simple element idealizations. At these times, the force method was quite popular as its solution directly provided the stresses/forces of the structure. Further, the crucial step of this method, namely, the classification of independent and redundant forces of the model was provided by the analyst himself with his visual inspection. There were no generic analytical approaches available for their classification. In the post computer era, analysts wanted to take the assistance of the computers and solve complex finite element models for their stresses and forces. Naturally, the choice of the method of analysis becomes dependent on how simple it is to implement on computers and generic it is, to consider complex finite element models. Force method could not compete on this account with the displacement method as there were no simple suitable automated algorithms even to classify the forces. This aspect literally drove back the force method into the background of structural analysis and brought forward the displacement method to gain its current popularity.

Some research efforts are still being made to revive the past popular force method. They are focused on removing the stumbling block of the classification of its unknown forces. Two types of approaches were attempted and they are classifying the forces by analytical computations and the other is totally avoiding the need for identification. Various analytical approaches are proposed to identify the redundant forces and many of them can consider only a specific type of structure and their generalization to complex structures is still being pursued. They can be categorized under Topological Force, Graph theory, Algebraic Force and Mixed Algebraic-Combinatorial Force Methods[1]. Topological methods were developed by Henderson[2] and later they are generalized to skeletal structures[3]. Kaveh[4]

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proposed methods from Graph theory that applied cycle bases to different types of skeletal structures[5]. Among algebraic force methods, some important contributions worth mentioning are Denke's Structure Cutter method [6], Robinson's Eigenforce Method [7], Topcu's Turn Back LU decomposition method [8] and Kaneko's QR Procedure [9]. Pothen demonstrated the use of mixed algebraic-topological methods as well[10]. We observe all these methods are not generic and simple enough to consider current finite element models of complex structure. They need further developments before they can be exploited in software packages. On the other hand, the Integrated Force Method was proposed by Patnaik [12] which solves the forces by skipping the need for their classification. In the wake of the dominance of the current displacement method, it did not get the attention it deserved. Some of the related research and its versatility are presented in references [12-18].

The Integrated Force Method brought a generic approach to solve the forces of a complex structure but it left the task halfway by not classifying the independent and redundant forces. The classification of forces actually provides an easy path to formulate advanced investigations on dynamics, stability, and optimizations of structures. An appended analytical simple approach to classifying the forces with the integrated force method can bring it at par with the classical force method. Such an approach makes the force method as generic and implementable as the popular displacement method. At the same time, it allows us to have the advantage of directly relating the forces with applied loads. May be still the analyst is looking forward to having a force method that includes automated classification of forces. Perhaps this can pave the force method as competitive as displacement method in the times to come.

In this paper, the Integrated Force Method is extended to include the automated classification of independent and redundant forces. The algorithm proposed is very simple and generic and it can consider any complex structure idealized with different elements. While performing the classification of forces, it yields matrix relations between the displacements and independent forces and also between redundant and independent forces. Its implementation with computers is as simple as the displacement method. Great potential exists for its direct exploitation with general-purpose software packages for performing finite element analysis by force method.

2. Force Method with Automated Classification of Forces

The Force Method with automated classification of forces is formulated by extending the Integrated Force Method to include the automatic identification of independent and redundant forces. Thus, it brings the proposed formulation in line with the classical forces method. Apart from providing a reduced-order equilibrium equations, it not only provides matrix relations between the displacements and the independent forces but also the relation between redundant forces and independent forces. The steps to perform classification of forces are extremely simple and easily amenable for their implementation on computers. The built-in routines of computer software can be readily utilized to obtain the solution. The steps to perform the classification of forces starts with the deformation displacement relation of the Integrated Force Method.

2.1 Deformation Displacement Matrix Relation.

In the theory of elasticity, the differential equations of equilibrium are supplemented by a set of compatibility conditions to obtain the elasticity solution. Similarly, we can formulate them for a finite element model with the consideration of energy theorems.

The equilibrium equations of the structure, idealized with finite elements, relate its internal elemental forces with the applied external forces at the nodes. Generally, the number of unknown elemental forces is greater than the available number of equilibrium equations. The additional equations required to solve for the forces in the elements can be generated by the consideration of deformations of the elements compatible with its nodal displacements.

Consider an idealized finite element model of a general structure with $'m'$ number of elemental forces and $'n'$ number of nodal displacements. The equilibrium of elemental forces $\{F\}$ with applied nodal loads $\{P\}$ of a structure with $'\,k\,'$ elements can be written as

$$\{P\} = [B]\{F\} \tag{1}$$

where $[B]$ represents the equilibrium matrix of the structure of size $(m \times n)$. The $[B]$ is formed by assembling the elemental equilibrium matrices $[B_e]$. Generally, $m > n$ which necessitates additional $(m - n)$ equations to solve equation (1).

The integrated Force method proposed by Patnaik[15] developed these additional equations from the Deformation Displacement Relations(DDR). They are

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developed on the basis of energy theorems. The internal energy stored in the elements of the structure is equal to the work done by the externally applied forces.

The elastic deformations of a i-th element of a structure due to its elemental forces can be written

$$\{\beta_i\} = [G_i]\ \{F_i\} \tag{2}$$

where $\{\beta_i\}$, $[G_i]$ and $\{F_i\}$ represent deformation vector, flexibility matrix and force vector of the i-th element respectively. The equation (2) corresponding to the global structure is obtained by assembling the elemental matrices of the structure and is given by equation (3) as

$$\{\beta\} = [G]\ \{F\} \tag{3a}$$

where

$$\{\beta\} = \{\{\beta_1\}\ \{\beta_2\}\ \dots\dots\dots\dots\dots\dots \{\beta_k\}\} \tag{3b}$$

$$\{F\} = \{\{F_1\}\ \{F_2\}\ \dots\dots\dots\dots\dots\dots \{F_k\}\} \tag{3c}$$

$$[G] = \begin{bmatrix} [G_1] & 0 & \dots & 0 \\ 0 & [G_2] & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & [G_k] \end{bmatrix} \tag{3d}$$

The internal energy I_E of the structure can be written as

$$I_E = (1/2)\{\beta\}^T\{F\} \tag{4}$$

The work by the external loads $\{P\}$ for the nodal displacements $\{U\}$ of the structure is given by

$$W = (1/2)\{P\}^T\{U\} \tag{5}$$

Equating the internal energy with the work done and later substituting the equation (1) will lead to

$$\{P\}^T\{U\} = \{F\}^T\{\beta\} = \{F\}^T[B]^T\{U\} \tag{6}$$

Equation (6) can also be rewritten as

$$\{F\}^T[\ \{\beta\} - [B]^T\{U\}\ ] = 0 \tag{7}$$

As $\{F\} \neq 0$ for a loaded structure, it leads to the Deformation Displacement Relation given by equation (8) which relates the deformations of the elements with the displacements of the structure.

$$[\ \{\beta\} - [B]^T\{U\}\ ] = 0 \quad \text{or} \quad \{\beta\} = [B]^T\{U\} \tag{8}$$

2.2 Compatibility equations of Integrated Force Method

We observe equation (8) has m equations to solve n unknown nodal displacements. The system of equations is overdetermined and the solution for $\{U\}$ can

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only be obtained in terms of m deformations subjected to $(m-n)$ constraints among themselves. They are called Compatibility Condition and are obtained by solving equation (8). Resulting compatibility conditions can be symbolically written in a matrix form as

$$[C_\beta]\{\beta\} = 0 \tag{9}$$

where $[C_\beta]$ represents the matrix with the coefficients associated with the deformations of the structure. Equation (9) can also be written in terms of forces using equation (3) as

$$[C_\beta][G]\{F\} = 0 \tag{10}$$

Combining the equation (1) and (10) leads to the governing equation of the integrated force method as

$$\begin{bmatrix} [B] \\ [C_\beta][G] \end{bmatrix} \{\mathrm{F}\} = \begin{Bmatrix} \{\mathrm{P}\} \\ \{0\} \end{Bmatrix} = \{\mathrm{P}^*\} \tag{11}$$

The equation (11) has m equations to solve m forces and thus it provides solutions for forces without the need for classification of the forces of the structure.

2.3 Automated Classification of Forces (ACF)

The deformation displacement equation (8) is further extended to develop the method for Automated Classification of Forces. The method allows us to identify the independent and redundant forces while computing the solution for equation (8). The proposed automated procedure is quite simple and easy to implement on the computer.

For convenience, the equation (8) can be written in terms of Force Displacement Relations given by equation (12) by substituting equation (2) into the equation (8)

$$\{F\} = [G]^{-1}[B]^T\{U\} = [G_{Bt}]\{U\} \tag{12}$$

where $[G_{Bt}] = [G]^{-1}[B]^T$. Solving equation (12) for n displacements in terms of m forces leads to a solution with $(m-n)$ force constraint relations. One of the simple ways solving equation (12) is by performing Reduced Row Echelon Form (RREF)[19] on its augmented matrix. The augmented matrix $[A]$ of equation (12) can be expressed as

$$\big[[G_{Bt}]\ \ [I]\big]\begin{Bmatrix} \{U\} \\ \{F\} \end{Bmatrix} = [A]\begin{Bmatrix} \{U\} \\ \{F\} \end{Bmatrix}; \qquad [A] = \big[[G_{Bt}]\ \ [I]\big] \tag{13}$$

After performing reduced row echelon form on $[A]$, the resulting $[E]$ matrix can be expressed in a partitioned form as given below with the partition sizes indicated as

$$\begin{bmatrix} [E_{11}] & [E_{12}] \\ [E_{21}] & [E_{22}] \end{bmatrix} = \begin{bmatrix} [I]_{(n\times n)} & [S_F]_{(n\times m)} \\ [0]_{((m-n)\times n)} & [C_F]_{((m-n)\times m)} \end{bmatrix} \tag{14}$$

Observing the partitions of $[E]$, we find $[E_{11}]$ is a unit matrix and $[E_{21}]$ is a null matrix. The solution of equation (13), now can be written as

$$\{U\} = -[S_F]\{F\} \tag{15}$$

$$[C_F]\ \{F\} = \{0\} \tag{16}$$

Equation (15) relates the displacements with the elemental forces and the equation (16) describes the constraints among the elemental forces.

Now observe carefully the matrices $[S_F]$ and $[C_F]$. We find some clues to identify redundant and independent forces. We find lots of null columns in $[S_F]$ and they arise due to the substitution of the redundant forces in terms of independent forces while performing the reduced row echelon form. Hence the forces corresponding to null columns in $[S_F]$ can be identified as redundant forces. The forces assigned to non-null columns are identified as independent forces. In this way, one can easily identify the redundant and independent forces by observing the structure of the columns of $[S_F]$. This procedure is very simple and applicable to any finite element model idealization. Compacting equation (15) by removal of null columns leads a matrix $[S_{UI}]$ which directly relates the displacements with the independent forces. $\{F\}$ reduces to $\{F_I\}$ containing independent forces only. The resulting equation (15) becomes

$$\{U\} = [S_{UI}]\{F_I\} \tag{17}$$

The $[C_F]$ can be partitioned into two matrices one with columns identified with redundant forces and the other with independent forces. The matrix identified with redundant forces turns out to be a unit matrix. Call the partition identified with independent forces as $[C_{RI}]$. $\{F\}$ vector gets partitioned $\{F_R\}$ containing redundant forces only and $\{F_I\}$ containing the independent forces only. Now equation (16) can be rewritten as

$$[\ I\ ]\,\{F_R\} + [C_{RI}]\,\{F_I\} = \{0\} \text{ or } \{F_R\} = -[C_{RI}]\,\{F_I\} \tag{18}$$

Equation (18) directly relates redundant forces with the independent forces. Here RREF played a vital role in developing the Automated Force Classification.

In a given structure, one can form multiple sets of redundant forces and any one of them can be chosen to obtain the solution. The RREF computations are based on multiple manipulations on the rows of the auxiliary matrix. Depending on the row manipulation sequences utilized during reduction, it forms its own selection for choosing a set of redundant forces.

With the classification of elemental forces, the subsequent determination of the forces follows the classical style of force method of analysis. The equation (1) is written in terms of independent and redundant forces as

$$\{P\} = \left[[B_I] \;\; [B_R]\right] \begin{Bmatrix} \{F_I\} \\ \{F_R\} \end{Bmatrix} = [B_I]\{F_I\} + [B_R]\{F_R\} \tag{19}$$

where $[B]$ is partitioned according to independent and redundant forces as $[B_I]$ and $[B_R]$. Now substituting the equation (18) into equation (19) leads to reduced order equilibrium square matrix

$$[B_{Red}] = \left[[B_I] - [B_R][C_{RI}]\right] \tag{21}$$

Now forces and displacements of the structure are given by the equation (22) as

$$\{F_I\} = [B_{Red}]^{-1}\{P\} \quad ; \quad \{F_R\} = -[C_{RI}]\{F_I\} \quad ; \quad \{U\} = [S_{UI}]\{F_I\} \tag{22}$$

With computed forces in the elements, their stresses and strains can be computed utilizing the elements elastic parameters.

The automated classification of forces has brought the classical force method to an Automated Force method with an elegantly streamlined procedure. It does not need various mathematical tools mentioned earlier with the analytical methods. Its implementation for computers is as simple as the displacement method.

2.4 Elemental Equilibrium and flexibility Matrices

The force method allows developing their elemental equilibrium and flexibility matrices by assuming independent force and displacement distributions. However, care needs to be taken that the strains fields resulting from the assumed force distributions are similar to the strain fields resulting from the assumed displacement distributions.

In the illustrative examples, different types of structures are considered each idealized with different types of elements. Using principles of strain energy and complementary strain energy, the elemental equilibrium and flexibility matrices can easily be developed by substituting the assumed force and displacement fields. Elements with complex force distributions, it becomes very difficult to describe force fields with physically meaning full attributes. The distribution of forces is generally described through a set of force parameters that control the force distribution over the elements and they are called force degrees of freedom. Knowing the magnitudes of the force parameters, the resulting force distribution on the element can be captured.

For each illustrative example, the assumed force and displacement fields over the element are mentioned. They are used for the development of their elemental equilibrium and flexibility matrices.

2.5 Implementation Force Method on Computers

The implementation of the force method with automated classification of forces is quite simple and straightforward on the computer. Generally, there are six generic steps in its implementation. They are generation of elemental characteristic matrices, assembly of elemental matrices to obtain global matrices, setting up governing equations of equilibrium, solving the force-displacement equations, forming and solving the reduced equilibrium equations, and finally post-processing the results. The displacement method implementation follows similar lines except for the additional step of solving force-displacement equations.

The generic steps of implementation of force method with automated forces identification on computer are shown is given in figure 1

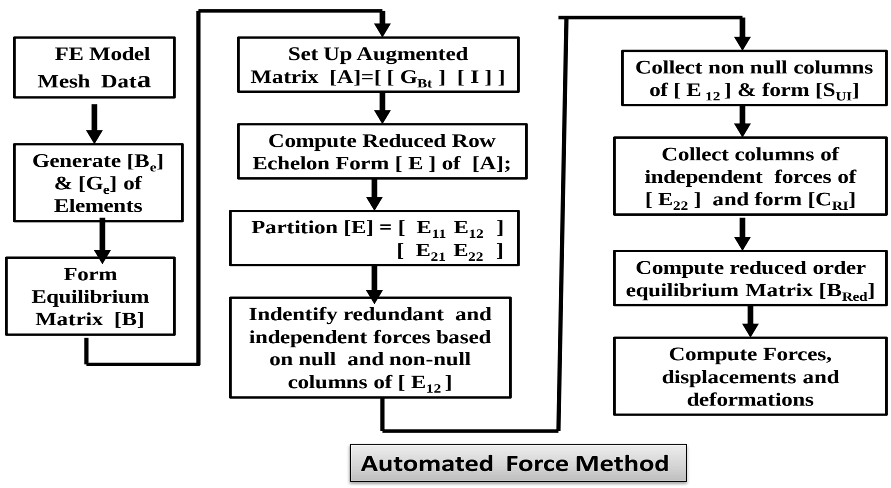

Figure 1. Schematic flow diagram of the Automated Force Method.

The input file of the finite model prepared for the displacement file can be directly used except changing the names of the elements. The computation of reduced row echelon form of the force-displacement equation can be made with built-in mathematical routines of the computer software packages. Simple post-processing of results provides the results for their direct interpretation. Force method now with automatic identification has all the ingredients for the development of a commercial analytical platform to perform finite element analysis of any complex structure.

3. Results and Discussions

The proposed force method with automated force identification is demonstrated successfully on finite element models idealized with various types of elements and their results are presented and discussed. These analyzed structures include such as truss, plane stress panel, rigid plane frames and plate in bending. The results of the automated redundant forces identification can be at best represented with the number of redundant forces identified with each element and their associated force degree of freedom.

To demonstrate the method of automated identification of forces in detail, a simple truss structure is considered and the various steps involved in the process of identification of redundant and independent forces are presented. For elements like plane stress, beam, and plate, the redundant forces are represented through generalized force degrees of freedom assigned with the force distribution functions. The generalized elemental forces and displacements are computed by solving the reduced force equilibrium equation. Computed forces and displacements are of any research value for this paper as our thrust is on the classification of forces. Hence the presentation of results is mainly focused on the automated identification of forces.

3.1 Automated Identification of Forces of a Simple Truss

The algorithm of automated force identification is demonstrated on a small truss structure. A truss structure with 8 nodes and 16 elements having multiple supports is considered and is shown in figure 2. The figure shows the numbering of nodes, elements and its 10 nodal displacement degrees of freedom. The truss element equilibrium matrix and the flexibility matrix are based on the axial linear displacement field and constant axial force in the element. Matrices $[S_F]$ and $[C_F]$ resulting from RREF operation are presented in Tables 1 and 2. They help us to visualize the automatic identification of independent and redundant forces.

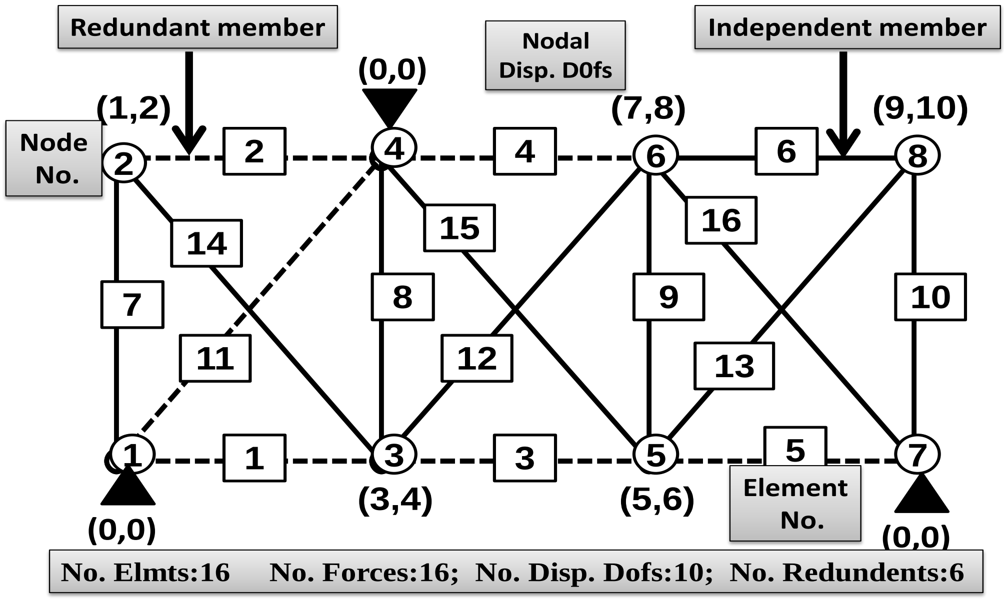

Figure 2. FEM Model of Truss with three supports and its Identified forces.

Observing closely the columns of $[S_F]$, we find some of its columns are null. The forces identified with these null columns are called redundant forces and they are (1,2,3,4,5,11) totaling to 6. The rest of the forces with non-null columns are called independent forces (6,7,8,9,10,12,12,14,15,16) totalling to 10. The number of independent forces identified corresponds to the number of displacement degrees of freedom as expected. Figure 2 also shows the identified independent and redundant members of the truss with solid black lines and dashed lines respectively. On the top of the matrices of Tables 1 and 2, they also show the original force numbering and the renumbered independent and redundant forces assigned with vector $\{F_I\}$ and $\{F_R\}$. When the columns of $[C_F]$ identified with redundant forces are put together, it turns out as a unit matrix. Matrix $[C_{RI}]$ is formed by collecting the columns identified with independent forces. The matrix $[C_{RI}]$ directly provides a relation between the redundant and independent forces given by equation (18). The truss structure identified with independent members should lead to a stable truss. The visual inspection of figure 2 confirms that the automated force method has done the classification of members appropriately. Thus, the reduced row echelon form on the auxiliary matrix has successfully performed the identification of forces and provided matrix relations required to compute the forces and displacements of the structure.

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Table 1. Partition [S_F] of RREF[A] of Truss Structure(Figure 2)

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16
	R1	R2	R3	R4	R5	I1	I2	I3	I4	I5	R6	I6	I7	I8	I9	I10
[S_F]=	0.00	0.00	0.00	0.00	0.00	0.00	1.33	0.00	0.00	0.00	0.00	1.67	0.00	1.67	0.00	1.67
	0.00	0.00	0.00	0.00	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	-0.00	0.00	-0.00	0.00	-0.00
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.33	0.00	0.00	0.00	1.67	0.00	0.00	0.00	1.67
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	-0.00	0.00	0.00	0.00	-0.00
	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	-1.33	1.33	0.00	0.00	-1.67	0.00	-3.33	-1.67
	0.00	0.00	0.00	0.00	0.00	-0.75	0.00	0.00	1.00	-1.00	0.00	-0.00	1.25	0.00	1.25	1.25
	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	1.33	0.00	0.00	-1.67	0.00	-1.67	0.00
	0.00	0.00	0.00	0.00	0.00	-0.75	0.00	0.00	0.00	-1.00	0.00	-0.00	1.25	0.00	1.25	1.25
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.33	0.00	0.00	-1.67	0.00	-1.67	0.00
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 2. Partition [C_F] of RREF[A] of Truss Structure(Figure 2)

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16
	R1	R2	R3	R4	R5	I1	I2	I3	I4	I5	R6	I6	I7	I8	I9	I10
[C_F]=	1.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.33	0.00	0.00	0.00	1.67	0.00	0.00	0.00	1.67
	0.00	1.00	0.00	0.00	0.00	0.00	-1.33	0.00	0.00	0.00	0.00	-1.67	0.00	-1.67	0.00	-1.67
	0.00	0.00	1.00	0.00	0.00	1.00	-0.00	1.33	-1.33	1.33	0.00	-1.67	-1.67	0.00	-3.33	-3.33
	0.00	0.00	0.00	1.00	0.00	1.00	0.00	0.00	0.00	1.33	0.00	0.00	-1.67	0.00	-1.67	0.00
	0.00	0.00	0.00	0.00	1.00	-1.00	0.00	0.00	1.33	-1.33	0.00	0.00	1.67	0.00	3.33	1.67
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00

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3.2 Identification of forces of different FE models

The success of the method of identification of forces is demonstrated on finite element models idealized with different types of elements namely axial bar element, plane stress element, beam element, and the plate element. Each model considers a structure with a large number of elements. The results show the number of identified redundant forces in terms of force degrees of freedom used to describe the force field distribution within each of the elements.

3.2.1 Large size Truss

A large truss with 10 horizontal bays and 5 vertical bays is considered and it is shown in figure 3. It has 215 elements and 128 displacement degrees of freedom resulting in 87 redundant members. Automated force identification has successfully identified the independent and redundant members and these results are shown in figure 3. The dotted lines show the identified redundant truss members while the solid black lines show identified independent members. Visual observation of the truss formed with the independent members is found to be stable.

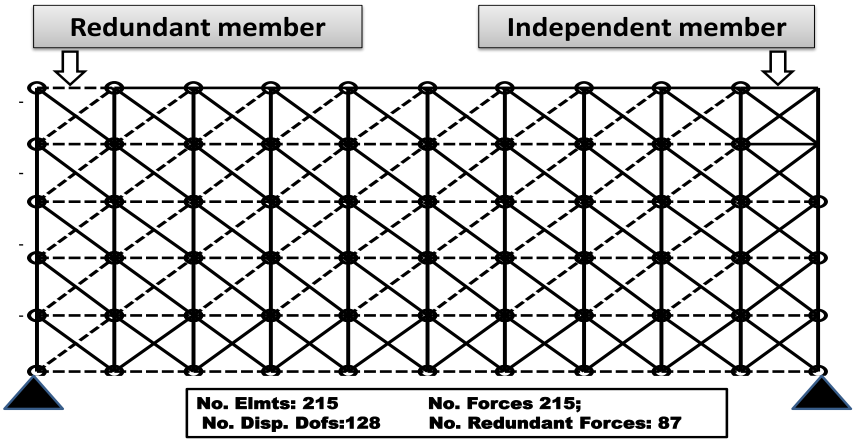

Fig. 4 Multi-bay truss with identified redundant and independent members

3.2.2 Plane Stress Problem

A plane stress problem of a thin panel with in-plane loads is considered with 50 plane stress elements and is shown in figure 4. It has 250 elemental forces and120 displacement degrees of freedom leading to a structure with 130 redundant forces. The plane stress element equilibrium and flexibility matrices are based on the assumed shape functions of displacement and force fields given by equations (23, 24, 25). The resulting plane stress element has 8 displacement degrees of freedom $(u_1.....u_4, v_1.....v_4)$ and the force field is described by 5 force freedom parameters$(\ f_1, f_2, f_3, f_4, f_5\ )$.

$$u = \frac{1}{4}\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)u_1 + \left(1+\frac{x}{a}\right)\left(1-\frac{y}{b}\right)u_2 + \left(1+\frac{x}{a}\right)\left(1+\frac{y}{b}\right)u_3 + \left(1-\frac{x}{a}\right)\left(1+\frac{y}{b}\right)u_4\right] \tag{23}$$

$$v = \frac{1}{4}\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)v_1 + \left(1+\frac{x}{a}\right)\left(1-\frac{y}{b}\right)v_2 + \left(1+\frac{x}{a}\right)\left(1+\frac{y}{b}\right)v_3 + \left(1-\frac{x}{a}\right)\left(1+\frac{y}{b}\right)v_4\right] \tag{24}$$

$$N_x = f_1 + f_2\frac{y}{b}\ ; \quad N_y = f_3 + f_4\frac{x}{a}\ ;\ N_{xy} = f_5\ ; \tag{25}$$

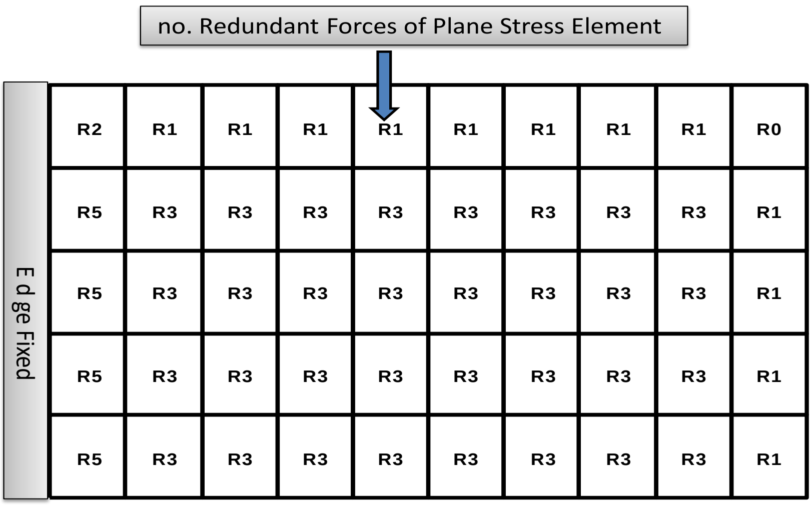

Figure. 4 Plane Stress Structure with identified redundant and independent forces

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Automated Force Method successfully identified independent and redundant forces of the panel. Unlike truss structure, each element can have a maximum of 5 identified redundant forces. Figure 4 also shows the number of redundant forces identified with each element at its center. Table 3 provides a break down between the number of elements against their identified number of redundant forces of the element. There are 4 elements with all of its forces that are considered redundant forces and they can be called redundant elements. Unless all the redundant forces lead to redundant elements, the visual observation of a stable structure cannot be made. In this case, we cannot make any such observations.

Table 3. Number of Redundant forces identified with elements (Plane Stress)

Redundant forces of an element	0	1	2	3	4	5
No. of such identified elements	1	12	1	32	0	4

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3.2.3 Rigid Plane Frame Structure

A rigid 2D-plane frame (10x5) bays is shown in figure 5. It has 105 elements, 315 elemental forces and165 displacement degrees of freedom. It has 155 redundant forces. The 2D-beam element has 3 displacement degrees of freedom at each of its node namely, axial displacement, transverse displacement, and rotation. The

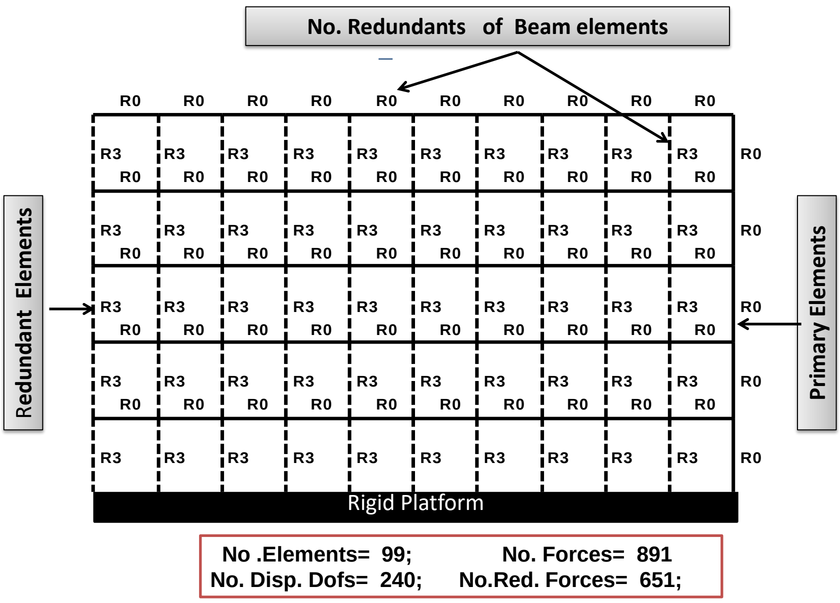

Figure 5. Plane Frame problem showing identified redundant forces in the frame elements

shape function described by equations (26) to (28). The axial forces and bending moment are described 3 force degrees of freedom given by equation (29). These equations are used to compute its element's equilibrium and flexibility matrices. The element has 6 displacement degrees of freedom and 3 force degrees of freedom.

$$u = \left(1 - \frac{x}{a}\right) u_1 + \left(\frac{x}{a}\right) u_2 \tag{26}$$

$$w = N_1 w_1 + N_2 w_{1x} + N_3 w_2 + N_4 w_{2x}$$

$$N_1 = \frac{1}{2} - \frac{3x}{4a} - \frac{x^3}{4a^3}; \qquad N_2 = \frac{a}{4} - \frac{x}{4} - \frac{x^2}{4a} + \frac{x^3}{4a^2} \tag{27}$$

$$N_3 = \frac{1}{2} + \frac{3x}{4a} - \frac{x^3}{4a^3}; \qquad N_4 = \frac{a}{4} + \frac{x}{4} - \frac{x^2}{4a} + \frac{x^3}{4a^2} \tag{28}$$

$$N_x = f_1; \qquad M_x = f_2 + \frac{x}{a} f_2 \tag{29}$$

where $(u_1,\ u_2,\ w_1, w_{1x},\ w_2, w_{2x})$ are displacement degrees of freedom and (f_1, f_2, f_3). are force degrees of freedom.

The automated force method successfully identified the independent and redundant forces and its corresponding identification results are also shown in figure 5. Each element can have a maximum of 3 redundant forces identified. A pictorial view of the number of redundant forces identified with each element is shown above or side of the elements. The breakdown of a number of elements having the same number of redundant forces is presented in Table 4. It is interesting to observe that there are 50 redundant elements and the rest of the elements are independent elements. As all the redundant forces lead to redundant elements, it is easy to visualize from the figure that the structure with independent elements represents a statically determinate part of the structure.

Table 4. Number of Redundant forces identified with elements(Plane Frame)

Redundant forces of an element	0	1	2	3
No. of such elements	55	0	0	50

3.2.4 Plate Bending

The plate bending problem is considered with an element grid of (11x9) bays with all sides fixed is considered and is shown in figure 7. It has 99 elements, 891 elemental forces and 240 displacement degrees of freedom. It has 651 redundant forces. The plate bending element modal has transverse displacements and two rotations degrees of freedom at each of its nodes. A polynomial is used to represent the transverse displacement field. The bending moment distribution on the element are also represented by polynomials with force parameters ($f_1, f_2, f_3, f_4, f_5, f_6\ f_7, f_8, f_9$)

$$w = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + y + a_5 y^2 + a_6 x^3 + a_7 x^2 y$$
$$a_8 x\, y^2 + a_9 y^3 + a_{10} x^{\,3} y + a_{11} x\, y^3 \quad (30)$$
$$M_{xx} = f_1 + f_2 x + f_3 y + f_4 xy; \quad M_{yy} = f_5 + f_6 x + f_7 y + f_8 xy; \quad M_{xy} = f_9 \quad (31)$$

The resulting plate bending element equilibrium matrix will have 12 displacement degrees of freedom and 9 force degrees of freedom. The proposed algorithm was applied to identify the independent and redundant forces and it successfully identified

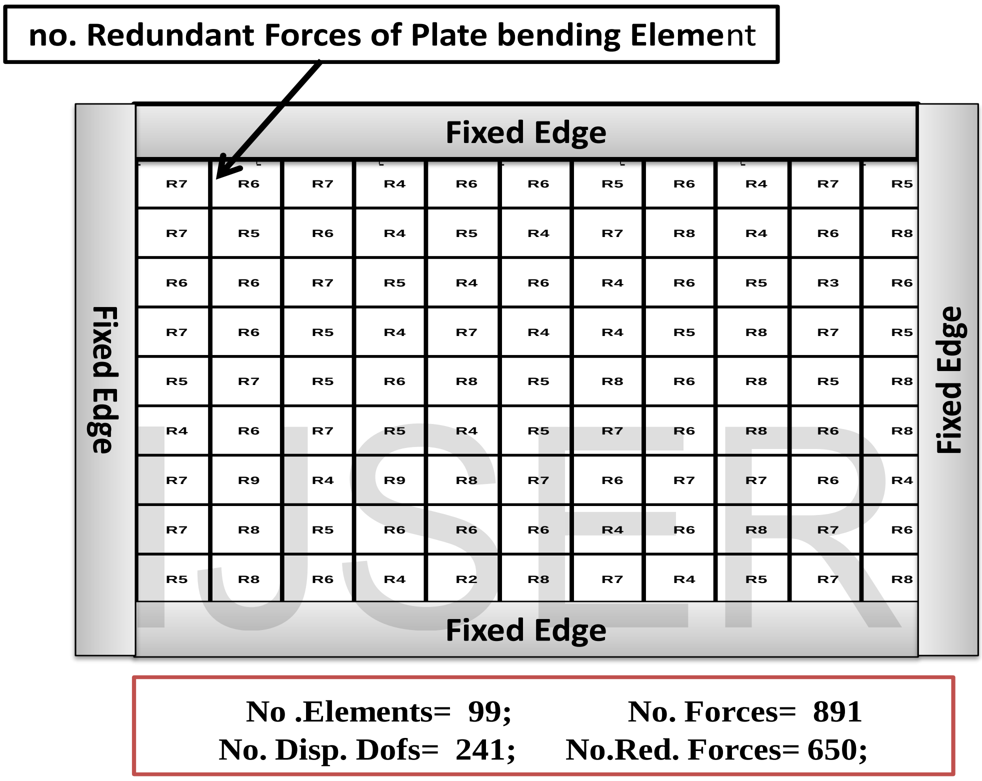

Figure 6. Plate in bending with identified redundant forces in the elements

all the redundant forces and the results are shown in figure 6. It provides a pictorial view of the number of identified redundant forces of each element by displaying it at its center. The breakdown of the number of redundant forces with their corresponding number of such elements is presented in Table 5. We observe that there are 8 redundant elements. The automated force method successfully identified the redundant forces of the model.

Table 5. Number of Redundant forces identified with elements (Plate Bending)

Redundant forces of an element	0	1	2	3	4	5	6	7	8	9
No. of such elements	0	0	0	1	4	15	28	27	16	8

4. Conclusions

Force method was continuously challenging the analysts on the analytical identification of redundant and independent forces of a structure modeled with finite elements. This problem was successfully resolved by the proposed algorithm of automated identification of forces of any complex structure modeled with finite elements. The algorithm is extremely simple and easy to implement on computers. Making use of the force-displacement equations of the integrated force method, the automated force identification algorithm was developed.

The solution of force-displacement equations allows easily to identify the independent and redundant forces by a simple look up at the structure of its matrix equations. Further, it develops three matrix relations between the displacements and forces, redundant and independent forces and finally a square matrix of reduced equilibrium equation.

The set of three equations together plays an important role in the force method in extending it to perform advanced structural investigations in the fields of statics, dynamics, stability and optimization with ease unlike in displacement method where they need significant additional numerical computational effort.

The force method is brought to a level where it can be implemented on a computer with as much ease as that of the displacement method. Now proposed algorithm is ready for its exploitation in the commercial finite element packages based on force method.

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